Interaction morphisms

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Motivation

- What is a systematic way to go about running effectful computations (in functional programming), handling effects, reducing effects to manipulation of state?

- U. (MFPS 2015): stateful runners.
  The theory is centered around associating to a monad a comonad going via the (generally large) Lawvere theory corresponding to the monad.

- This talk: interaction morphisms as a more abstract approach.

- On a higher-level, this is a functional programmer's take on certain types of protocols of two-party communication (must be closed under sequential composition of sessions).

This talk

- Interaction morphisms
  as an abstract way to specify environments capable of handling effects in computations and how they do it
- Their relationship to runners of effects . . .
- . . . and to monad morphisms

Interaction morphisms: Examples

- $TX = S \Rightarrow S \times X$, $DY = S \times (S \Rightarrow Y)$

$$\theta_{X,Y} : \underbrace{(S \Rightarrow S \times X)}_{TX} \times \underbrace{(S \times (S \Rightarrow Y))}_{DY} \to X \times Y$$

- $TX = S \Rightarrow S \times X$, $DY = C \times (C \Rightarrow Y)$
  in the presence of get : $C \to S$, put : $C \times S \to C$
  satisfying the *lens* laws

$$\theta_{X,Y} : \underbrace{(S \Rightarrow S \times X)}_{TX} \times \underbrace{(C \times (C \Rightarrow Y))}_{DY} \to X \times Y$$
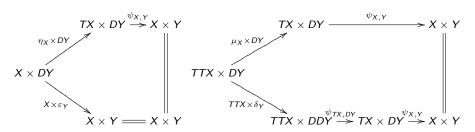
- $TX = \mu Z. X + \Sigma s : S.(P\, s \Rightarrow Z)$,
  $DY = \nu Z. Y \times \Pi s : S.P\, s \times Z$

Interaction morphisms

- Given a monad $T = (T, \eta, \mu)$ and a comonad $D = (D, \varepsilon, \delta)$ on a category with finite products (or, more generally, a monoidal category).
- An *interaction morphism* between T, D is a nat. transf. ψ with comps.
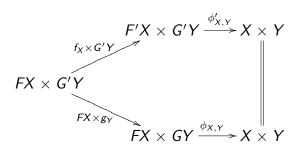
$$\psi_{X,Y} : TX \times DY \to X \times Y$$

satisfying

Interaction morphisms as monoids

- Interaction morphisms are monoids in a suitable monoidal category (just as monads, comonads).
- An object in this category is
  a pair of functors F, G, with a nat. transf. ϕ with comps.
  $\phi_{X,Y} : FX \times GY \to X \times Y$
- A map between (F, G, ϕ), (F', G', ϕ') is a pair of nat. transfs. $f : F \to F'$, $g : G' \to G$ such that

$$
\begin{array}{ccc}
& F'X \times G'Y \xrightarrow{\phi'_{X,Y}} X \times Y \\
\nearrow^{f_X \times G'Y} & & \big\| \\
FX \times G'Y & & \\
\searrow_{FX \times g_Y} & & \\
& FX \times GY \xrightarrow{\phi_{X,Y}} X \times Y
\end{array}
$$

Runners

- Given a monad T on a category $\mathcal{C}$.
- A *runner* of T is an object Y with a nat. transf. θ with comps.

$$\theta_X : TX \times Y \to X \times Y$$

satisfying

$$
\begin{array}{ccc}
TX \times Y \xrightarrow{\theta_X} X \times Y & \quad & TX \times Y \xrightarrow{\hspace{4cm}\theta_X\hspace{4cm}} X \times Y \\
\eta_X \times Y \big\uparrow \quad\quad \big\| & \quad & \mu_X \times Y \big\uparrow \quad\quad\quad\quad\quad\quad\quad \big\| \\
X \times Y =\!=\!=\!= X \times Y & \quad & TTX \times Y \xrightarrow{\theta_{TX}} TX \times Y \xrightarrow{\theta_X} X \times Y
\end{array}
$$

- More concisely, a runner of a monad T is an object Y together with a monad morphism from T to the state monad for Y.

$$\frac{TX \times Y \to X \times Y}{TX \to \underbrace{Y \Rightarrow X \times Y}_{\mathsf{s}^Y X}}$$

Interaction morphisms and runners

- Interaction morphisms between T, D are in a bijection with carrier-preserving functors from coalgebras of D to runners of T.

$$(i(\psi))_X^{Y,\gamma} = TX \times Y \xrightarrow{TX \times \gamma} TX \times DY \xrightarrow{\psi_{X,Y}} X \times Y$$

$$(i^{-1}(\theta))_{X,Y} = TX \times DY \xrightarrow{\theta_X^{DY,\delta_Y}} X \times DY \xrightarrow{X \times \varepsilon_Y} X \times Y$$

Interaction morphisms and monad morphisms

- Given a comonad D on $\mathcal{C}$, we can turn it into a monad $\ulcorner D \urcorner$ by

$$\ulcorner D \urcorner X = \int_Y DY \Rightarrow X \times Y$$

(because $\ulcorner - \urcorner : [\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$ is lax monoidal, hence sends monoids to monoids)

- Interaction morphisms between T, D are in a bijection with monad morphisms between T and $\ulcorner D \urcorner$, i.e., nat. transfs. $\tau : T \to \ulcorner D \urcorner$ satisfying certain equations.

$$\frac{\theta_{X,Y} : TX \times DY \to X \times Y}{(\mathrm{cur}\,\theta)_X : TX \to \underbrace{\int_Y DY \Rightarrow X \times Y}_{\ulcorner D \urcorner X}}$$
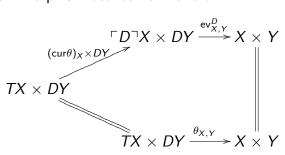
Interaction morphisms and monad morphisms

- The obvious natural transformation ev^D with components

$$\mathrm{ev}^D_{X,Y} : \underbrace{(\int_Y DY \Rightarrow X \times Y)}_{\ulcorner D \urcorner X} \times DY \to X \times Y$$

is an interaction morphism.

- The monad morphism $\mathrm{cur}\,\theta$ is the unique interaction morphism morphism between ev^D and θ.

Summing up

- Interaction morphisms seem (from the categorical point of view) a natural concept with neat properties.
- They also seem to be a good abstraction for analyzing running/handling of effects.
- Alternatively, they are way to talk about communication protocols of two parties over a channel and the duality involved.
- Lots of cool category theory still to be worked out.