Recall for free:

preorder-respecting state monads in F*

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- An effectful dependently-typed functional language

$$\texttt{a,b ::= ...} \mid \texttt{x:a} \rightarrow \textbf{PURE}\ \texttt{b}\ \texttt{wp}_{\texttt{p}}$$

$$\mid \texttt{x:a} \rightarrow \textbf{DIV}\ \texttt{b}\ \texttt{wp}_{\texttt{d}}$$

$$\mid \texttt{x:a} \rightarrow \textbf{STATE}\ \texttt{b}\ \texttt{wp}_{\texttt{s}}$$

- An effectful dependently-typed functional language

a,b ::= ... | x:a → **PURE** b wp_p

| x:a → **DIV** b wp_d

| x:a → **STATE** b wp_s

PURE, DIV, STATE - Dijkstra monads

- An effectful dependently-typed functional language

`a,b` **weakest precondition predicate transformers**

`| x:a → DIV b` wp_d

`| x:a → STATE b` wp_s

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`| x:a → ` **`STATE`** `b` wp_s

PURE, DIV, STATE - Dijkstra monads

- Some resources:
  - www.fstar-lang.org
  - "Dependent Types and Multi-Monadic Effects in F*" [POPL'16]
  - "Dijkstra Monads for Free" [POPL'17]

Outline

- A recurring phenomenon
- Preorder-respecting (Dijkstra) state monads F*
- Some examples
- A glimpse of the formal metatheory
- What are Dijkstra monads category theoretically?

(if time permits)

A recurring phenomenon

Example 1

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```
let s   = get () in
let _   = put (s + 1) in
let s'  = get () in
f () ;
let s'' = get () in
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;

- How to prove the 2nd assert "for free"?
- How to avoid global spec. in the type of f about $s' \le s''$?
- Generalise to other preorders and stable predicates?

Example 2

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```
val f : ref int → ST unit (fun s0 → True)
                          (fun s0 _ s1 → True)

let f r =
  let r' = alloc 0 in
  g r r'
```

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```
FStar.ST.recall r ;
```

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let f r =
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```

```
FStar.ST.recall r ;
```

- FStar.ST.recall is used pervasively in practice
- Can't implement it - is taken as an axiom
- It is intuitively correct - there is no dealloc op. in F*
- How to make this intuition formal?

```
> r') ;
```

Example 3

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Monotonic references in `FStar.Monotonic.RRef`

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type m_ref (reg:rid) (a:Type) (rel:preorder a)
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Provides operations

- `recall` - works as in `FStar.ST.recall`
- `witness` - witness a predicate holding value of a ref.
- `testify` - a previously witnessed predicate holds for a ref.

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also has to be taken as an axiom

Used pervasively in `mitls-fstar`

- for monotone sequences, -counters and -logs

State monads in

State monads in

State monads in F*

The state monad in F* has (roughly) the following type

```
STATE :  a:Type
      → wp:((a → state → Type0) → state → Type0)
      → Effect
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WPs of state operations are familiar from Hoare Logic, e.g.

```
val put : x:state
        → STATE unit (fun p s → p () x)
```

Preorder-respecting state monads in F*

High-level picture

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- introduce a ■-modality on stable predicates

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↑

"witnessed"

Relations and predicates

Relations and predicates

Relations and preorders

```
let relation a = a → a → Type₀

let preorder a = rel:relation a
  { (forall x     . rel x x) ∧
    (forall x y z . rel x y ∧ rel y z ⇒ rel x z) }
```

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```

Predicates and stability

```
let predicate a      = a → Type₀

let stable_p #a rel = p:predicate a
  { forall x y . p x ∧ rel x y ⇒ p y }
```

PSTATE and PST

PSTATE and PST

The signature of preorder-respecting state monads

```
PSTATE : rel:preorder state
       → a:Type
       → wp:((a → state → Type₀) → state → Type₀)
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PSTATE and **PST**

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We add **PSTATE** into the effect hierarchy of F* via **STATE**

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We add **PSTATE** into the effect hierarchy of F* via **STATE**

Note: Unfortunately, at the moment we can't define

```
sub_effect (forall state rel . Pure ⇝ PSTATE rel)
```

But we can make sub-effecting work for instances of **PSTATE**!

PSTATE and PST

The signature of preorder-respecting state monads

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PSTATE : rel:preorder state
        → a:Type
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```

Analogously to **STATE**, we again use syntactic sugar

```
PST : rel:preorder state
     → a:Type
     → pre:(state → Type₀)
     → post:(state → a → state → Type₀)
     → Effect
```

`get` and `put`

get and put

```
val get : #rel:preorder state
        → PST rel state (fun _ → True)
                        (fun s₀ s s₁ → s₀ = s ∧ s = s₁)
```

get and put

pre and post are exactly as for **STATE** and **ST**

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val put : #rel:preorder state
        → x:state
        → PST rel unit (fun s₀ → rel s₀ x)
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get and put

pre and post are exactly as for **STATE** and **ST**

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the change wrt. **STATE** and **ST**

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val put : #rel:preorder state
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We introduce an uninterpreted function symbol

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val ■ : #rel:preorder state
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We assume logical axioms, e.g., functoriality:

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forall p p' . (forall s . p s ⇒ p' s) ⇒ (■p ⇒ ■p')
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Two readings of ■p

p held at some past state of an **PSTATE** computation

p holds at all states reachable from the current with **PSTATE**

witness and recall

witness and recall

```
val witness : #rel:preorder state
            → p:stable_p rel
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val recall : #rel:preorder state
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```

Examples

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  - using `FStar.Heap.heap`
    (need a source of freshness for `alloc`)
  - ⋆ using our own heap type
    (source of freshness built into the heap)

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- Immutable references and other preorders
- Monotonic references
- ⋆ Temporarily ignoring the constraint on `put` via snapshots

Our `heap` and `ref` types

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The `heap` and `ref` types

```
let heap  = h:(nat * (nat → option (a:Type0 & a)))
              { ... }

let ref a = nat
```

Our heap and ref types

freshness counter

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both ops. have (r ∈ h) refinements on references

```
let ref a = nat
```

We can define sel and upd and gen_fresh operations

and prove expected properties, e.g.:

```
r <> r'  ⇒  sel (upd h r x) r' = sel h r'
```

Our heap and ref types

The heap and ref types

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let heap  = h:(nat * (nat → option (a:Type0 & a)))
             { ... }

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```

freshness counter

both ops. have (r ∈ h) refinements on references

Goal: use this heap as drop-in replacement for F*'s heap

(but in F*'s heap, sel and upd don't have (r ∈ h) refinements)

- change the type of refs. to (let ref a = nat * a)
- make use of the presence LEM in WPs for checking (r ∈ h)

Allocated references example

Allocated references example

The type of refs. and preorder for recalling allocation

```
let ref a     = r:(Heap.ref a){ ■(fun h → r ∈ h) }

let rel h₀ h₁ = forall a r . r ∈ h₀ ⇒ r ∈ h₁

AllocST a pre post = PST rel a pre post
```

Allocated references example

The type of refs. and preorder for recalling allocation

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let ref a      = r:(Heap.ref a){ ■(fun h → r ∈ h) }

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AllocST a pre post = PST rel a pre post
```

AllocST operations crucially use `witness` and `recall`, e.g.,

```
let read #a (r:ref a) =
  let h = get () in
  recall (fun h → r ∈ h) ;
  sel h r
```

Snapshots

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We first define snaphsot-capable state as

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The snaphsot-capable preorder is indexed by `rel` on `state`

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let s_rel (rel:preorder state) s₀ s₁ =
  match (snd s₀) (snd s₁) with
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let s_rel (rel:preorder state) s0 s1 =
  match (snd s0) (snd s1) with
  | None         None         ⇒ rel (fst s0) (fst s1)
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  | None       (Some s)   ⇒ rel (fst s₀) s
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  | (Some s)   None       ⇒ rel s (fst s₁)
```

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  | None        (Some s)    ⇒ rel (fst s0) s
  | (Some s)    None        ⇒ rel s (fst s1)
  | (Some s0') (Some s1') ⇒ rel s0' s1'
```

`read` and `write`

read and write

```
val read : #rel:preorder state
         → SST rel state
               (fun s₀ → True)
               (fun s₀ s s₁ → fst s₀ = s ∧ s = fst s₁ ∧
                              snd s₀ = snd s₁)
let write #rel x = ...
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val write : #rel:preorder state
          → x:state
          → SST rel unit
               (fun s₀ → s_rel rel s₀ (x,snd s₀))
               (fun s₀ _ s₁ → s₁ = (x,snd s₀))
let write #rel x = ...
```

witness and recall

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```
val witness : #rel:preorder state
            → p:stable_p rel
            → SST rel unit (fun s₀ → p (fst s₀) ∧
                                     snd s₀ = None)
                           (fun s₀ _ s₁ → s₀ = s₁ ∧ ■p)
let witness #rel p = ...
```

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let witness #rel p = ...

val recall : #rel:preorder state
           → p:stable_p rel
           → SST rel unit (fun s₀ → ■p ∧ snd s₀ = None)
                         (fun s₀ _ s₁ → s₀ = s₁ ∧
                                        p (fst s₁))
let recall #rel p = ...
```

snap and ok

snap and ok

```
val snap : #rel:preorder state
         → SST rel unit
             (fun s₀ → snd s₀ = None)
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let snap #rel = ...
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let snap #rel = ...

val ok : #rel:preorder state
      → SST rel unit
          (fun s₀ → exists s . snd s₀ = Some s ∧
                               rel s (fst s₀))
          (fun s₀ _ s₁ → fst s₀ = fst s₁ ∧
                         snd s₁ = None)
let ok #rel = ...
```

Example use of SST

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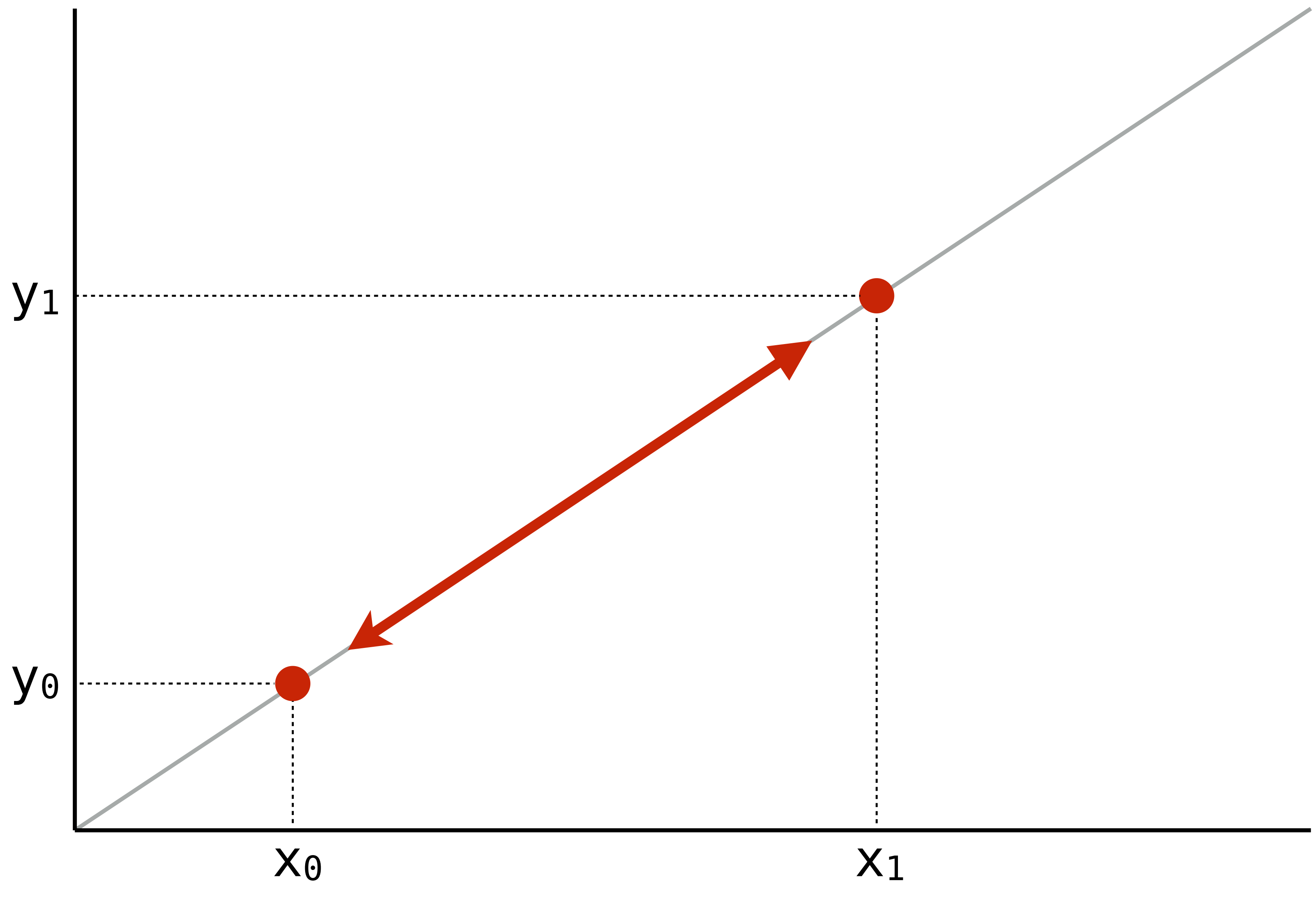

- Implementing a 2D point using two locations
- E.g., want to enforce that ● can only move along some line

A glimpse of the formal metatheory

PSTATE formally

PSTATE formally

We work with a small calculus based on EMF* from DM4F

```
t, wp, ::= state | rel | x:t1 → Tot t2 | x:t1 → PSTATE t2 wp | ...
e, φ       | x | fun x:t → e | e1 e2 | (e1,e2) | fst e | ...
           | return e | bind e1 x:t.e2
           | get e | put e | witness e | recall e
```

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Typing judgements have the form

```
G ⊢ e : Tot t
G ⊢ e : PSTATE t wp
```

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There is also a judgement for logical reasoning in WPs

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G | Φ ⊨ φ
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nat. deduction for classical predicate logic

Operational semantics

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Small-step call-by-value reduction relation

$$(\Phi, s, e) \longrightarrow (\Phi', s', e')$$

where

- Φ is a finite set of (witnessed) stable predicates
- `s` is a value of type `state`
- `e` is an expression

Operational semantics

Small-step call-by-value reduction relation

```
(Φ,s,e) ⟶ (Φ',s',e')
```

where

- Φ is a finite set of (witnessed) stable predicates
- s is a value of type `state`
- e is an expression

Examples of reduction rules

```
(Φ,s,put v)       ⟶ (Φ,v,return ())

(Φ,s,witness v)   ⟶ (Φ ∪ {v},s,return ())
```

Progress thm. for **PSTATE**

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$\forall$ `f t wp .`

$\vdash$ `f :` **`PSTATE`** `t wp`

$\Rightarrow$

1. $\exists$ `v . f = return v`

$\vee$

2. $\forall$ `Φ s .` $\exists$ `Φ' s' f' . (Φ,s,f)` $\longrightarrow$ `(Φ',s',f')`

Preservation thm. for **PSTATE**

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$$
\begin{array}{l}
\forall\ f\ t\ wp\ \Phi\ s\ \Phi'\ s'\ f'.\\
\quad \vdash f : \mathbf{PSTATE}\ t\ wp\ \ \wedge\ \ (\Phi,s)\ wf\ \ \wedge\\
\quad (\Phi,s,f) \longrightarrow (\Phi',s',f')\\
\quad \Rightarrow\\
\quad \forall\ post\ .\ \blacksquare\Phi \vDash wp\ post\ s\\
\qquad\qquad \Rightarrow\\
\qquad\qquad \Phi \subseteq \Phi'\ \ \wedge\ \ (\Phi',s')\ wf\ \ \wedge\\
\qquad\qquad \blacksquare\Phi \vDash rel\ s\ s'\ \ \wedge\\
\qquad\qquad \exists\ wp'\ .\ \vdash f' : \mathbf{PSTATE}\ t\ wp'\ \ \wedge\\
\qquad\qquad\qquad \blacksquare\Phi' \vDash wp'\ post\ s'
\end{array}
$$

Preservation thm. for **PSTATE**

$\forall$ f t wp Φ s Φ' s' f'.

⊢ f : **PSTATE** t wp ∧ (Φ,s) wf ∧

(Φ,s,f) ⟶ (Φ',s',f')

⇒

■Φ = ■(**fun** x → φ_1 x ∧ … ∧ φ_n x)

$\forall$ post . ■Φ ⊨ wp post s

⇒

Φ ⊆ Φ' ∧ (Φ',s') wf ∧

■Φ ⊨ rel s s' ∧

∃ wp' . ⊢ f' : **PSTATE** t wp' ∧

■Φ' ⊨ wp' post s'

The proof requires an inversion property (in empty context)

$$\frac{\models\ \blacksquare\varphi\ \Rightarrow\ \blacksquare\psi}{\models\ \texttt{forall}\ x\ .\ \varphi\ x\ \Rightarrow\ \psi\ x}\ (\blacksquare\text{ - inv})$$

We justify (■ - inv) via a cut-elimination in sequent calculus

- where we have a single derivation rule for ■

$$\frac{G \vdash \Phi_1 \qquad G \vdash \Phi_2 \qquad G, x \mid \Phi_1, \varphi_1\ x, \ldots, \varphi_n\ x \vdash \psi_1\ x, \ldots, \psi_m\ x, \Phi_2}{G \mid \Phi_1, \blacksquare\varphi_1, \ldots, \blacksquare\varphi_n \vdash \blacksquare\psi_1, \ldots, \blacksquare\psi_m, \Phi_2}$$

Future work: model theory of ■

Conclusion

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In this talk we covered:

- preorder-respecting state monads in F*
- their formal metatheory
- some of the examples of these monads

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- preorder-respecting state monads in F*
- their formal metatheory
- some of the examples of these monads

Ongoing and future work:

- change F*'s libraries to use **PSTATE**
- **PSTATE** in DM4F setting? (how to reify it safely?)
- model theory of ■
- categorical semantics of Dijkstra monads (rel. monads.)

Dijkstra monad T in CT?

Dijkstra monad *T* in CT?

Type formation rule for a Dijkstra monad

$$\frac{\Gamma \vdash t : \mathsf{Type} \qquad \Gamma \vdash wp : WP\ A}{\Gamma \vdash T\ \ t\ wp : \mathsf{Type}}$$

The unit of a Dijkstra monad

$$\frac{\Gamma \vdash e : t}{\Gamma \vdash \mathtt{return}\ e : T\ \ t\ \left(WP.\mathtt{return}\ e\right)}$$

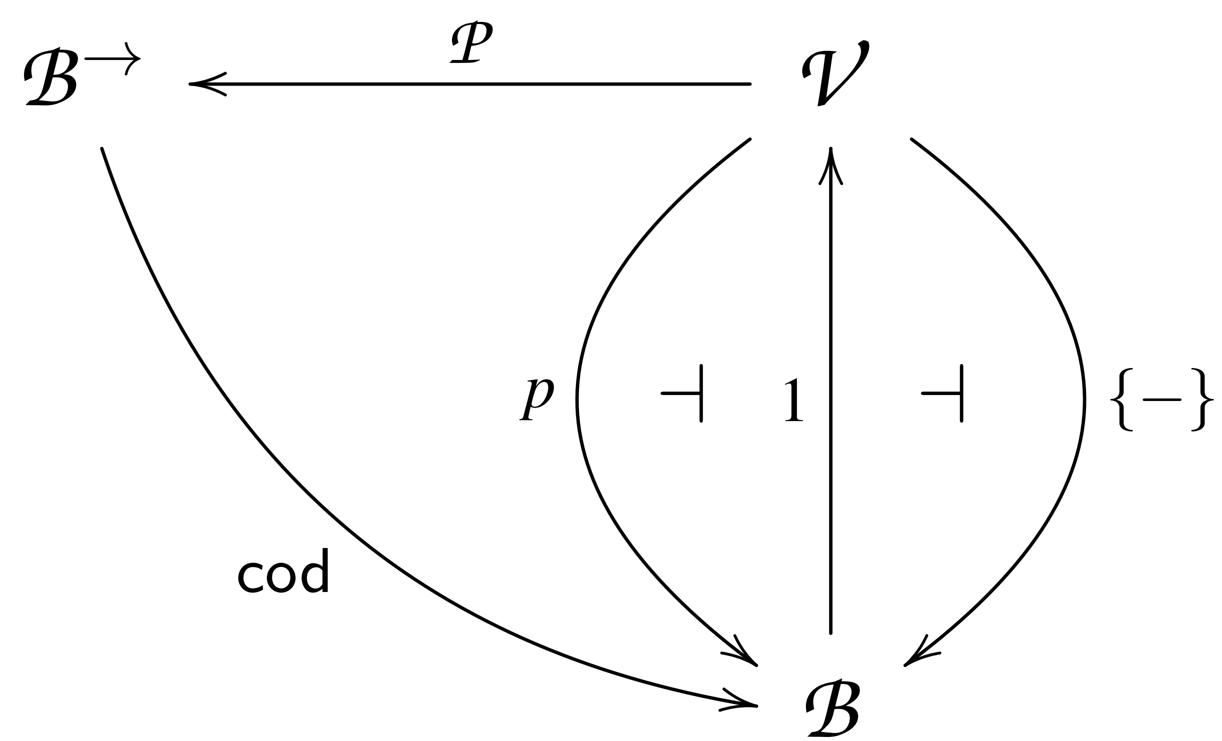
The Kleisli extension of a Dijkstra monad

$$\frac{\Gamma \vdash M : T\ t_1\ wp_1 \qquad \Gamma \vdash t_2 \qquad \Gamma, x{:}t_1 \vdash N : T\ t_2\ wp_2}{\Gamma \vdash \mathtt{bind}\ e_1\ x.e_2 : T\ \ t_2\ \left(WP.\mathtt{bind}\ wp_1\ x.wp_2\right)}$$

Dijkstra monad T in CT?

We'll work in the setting of closed comprehension cats., i.e.,

- $\mathcal{B}$ models contexts
- $\mathcal{V}$ models types in context
- terms in context Γ are modeled as global elements in $\mathcal{V}_{[\![\Gamma]\!]}$



- $\mathcal{P}$ is fully faithful

Dijkstra monad *T* in CT?

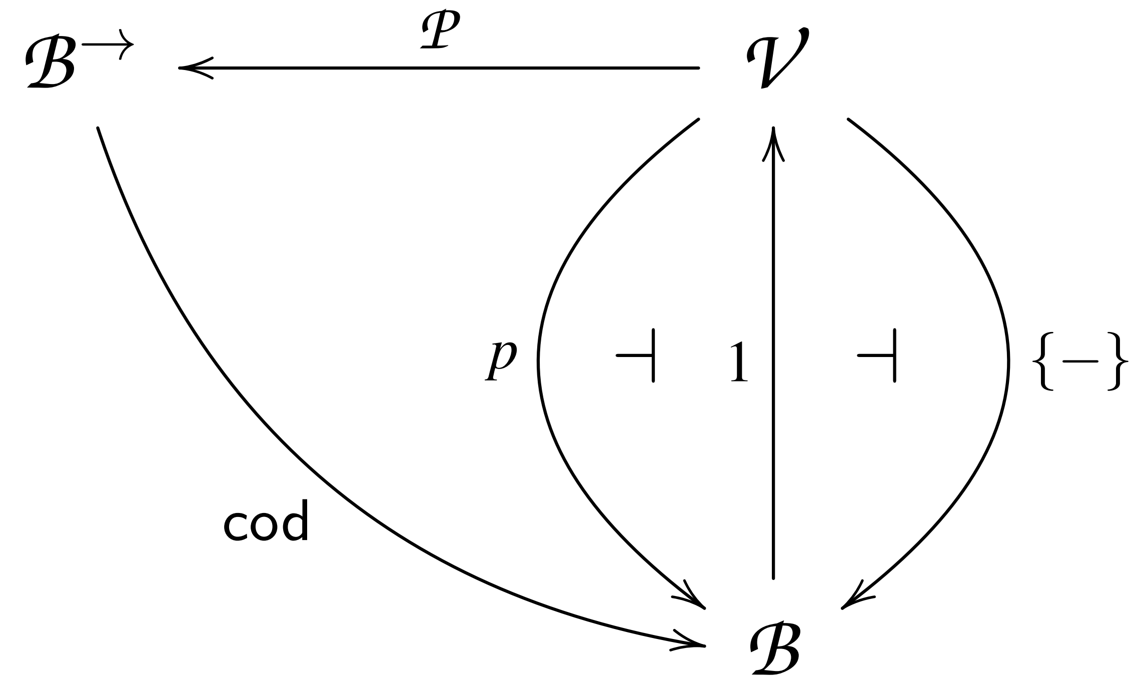
For modeling Dijkstra monads, we assume:

- a split fibred monad $WP : p \to p$
- a functor $T : \mathcal{V} \to \mathcal{V}$

s.t. $p \circ T = \{ - \} \circ WP$

T preserves Cartesian morphisms on-the-nose

$$\frac{\Gamma \vdash e : t}{\Gamma \vdash \texttt{return}\, e : T\ t\ (WP.\texttt{return}\ e)}$$



Can we model the unit and Kleisli ext. for *T* in known terms?

Dijkstra monad T in CT?

For modeling Dijkstra monads, we assume:

- a split fibred monad $WP : p \to p$
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dependency on WP

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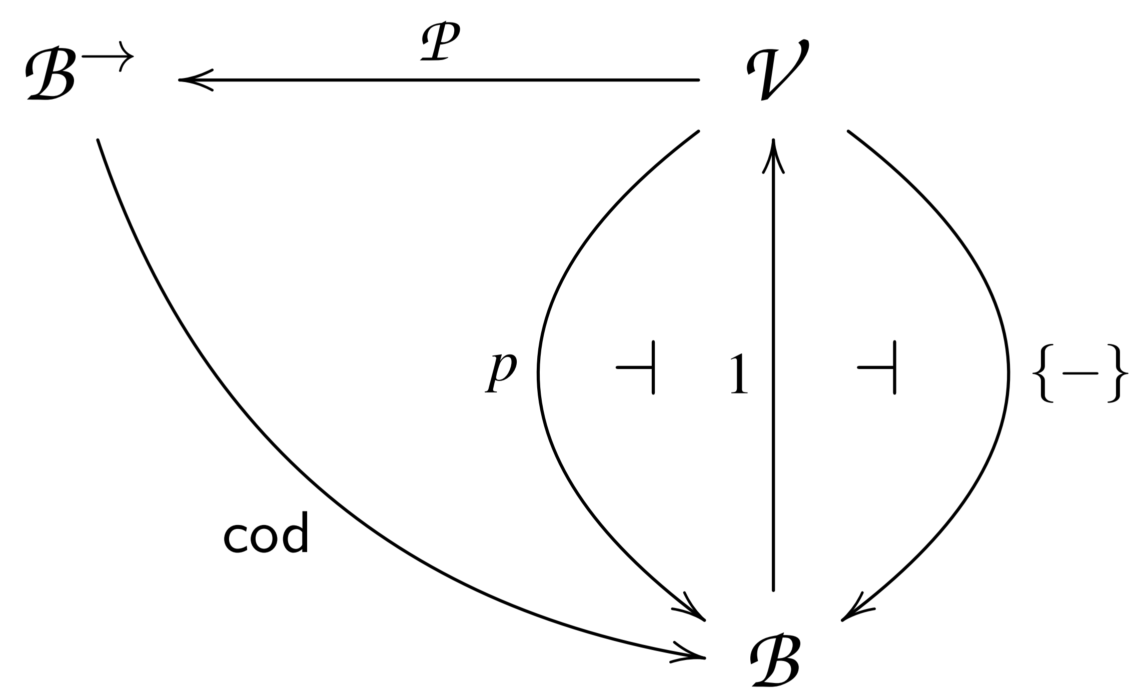

Can we model the unit and Kleisli ext. for T in known terms?

Dijkstra monad T in CT?

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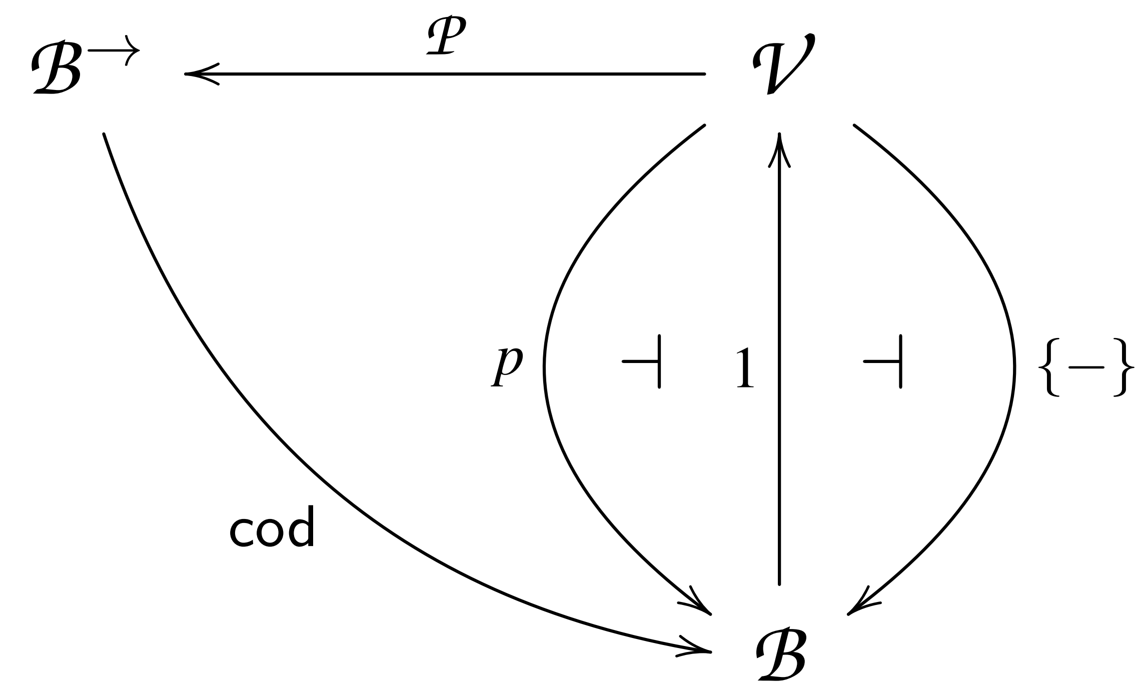
s.t. $p \circ T = \{ - \} \circ WP$

T preserves Cartesian morphisms on-the-nose

closed under substitution



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Can we model the unit and Kleisli ext. for T in known terms?

Dijkstra monad T in $\mathcal{B}^{\to}$

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The unit of a Dijkstra monad

$$\eta_A \quad : \quad \begin{array}{ccc} \{A\} & \xrightarrow{\qquad\qquad} & \{T(A)\} \\ {\scriptstyle \mathrm{id}_{\{A\}}}\downarrow & & \downarrow{\scriptstyle \pi_{T(A)}} \\ \{A\} & \xrightarrow[\{WP.\eta_A\}]{\qquad\qquad} & \{WP(A)\} \end{array}$$

Dijkstra monad T in $\mathcal{B}^{\to}$

The unit of a Dijkstra monad

$$\eta_A \quad : \quad \begin{array}{ccc} \{A\} & \longrightarrow & \{T(A)\} \\ {\scriptstyle \mathrm{id}_{\{A\}}}\downarrow & & \downarrow{\scriptstyle \pi_{T(A)}} \\ \{A\} & \xrightarrow[\{WP.\eta_A\}]{} & \{WP(A)\} \end{array}$$

The Kleisli extension of a Dijkstra monad

$$\left(\begin{array}{ccc} \{A\} & \xrightarrow{f} & \{T(B)\} \\ {\scriptstyle \mathrm{id}_{\{A\}}}\downarrow & & \downarrow{\scriptstyle \pi_{T(B)}} \\ \{A\} & \xrightarrow[\{g\}]{} & \{WP(B)\} \end{array} \right)^* \quad : \quad \begin{array}{ccc} \{T(A)\} & \longrightarrow & \{T(B)\} \\ {\scriptstyle \pi_{T(A)}}\downarrow & & \downarrow{\scriptstyle \pi_{T(B)}} \\ \{WP(A)\} & \xrightarrow[\{WP.(-)^*(g)\}]{} & \{WP(B)\} \end{array}$$

Dijkstra monad T in $\mathcal{B}^{\rightarrow}$

The unit of a Dijkstra monad

$$\{A\} \longrightarrow \{T(A)\}$$

> This data and the associated laws are precisely those for a relative monad
>
> $$\widehat{T} : \mathcal{V} \longrightarrow \overline{\mathsf{im}}(\{-\}) \downarrow \{-\}$$
>
> $$\widehat{T}(A) \stackrel{\text{def}}{=} \{T(A)\} \xrightarrow{\pi_{T(A)}} \{WP(A)\}$$
>
> on
>
> $$J : \mathcal{V} \longrightarrow \overline{\mathsf{im}}(\{-\}) \downarrow \{-\}$$
>
> $$J(A) \stackrel{\text{def}}{=} \{A\} \xrightarrow{\mathsf{id}_{\{A\}}} \{A\}$$

onad

$$\begin{array}{ccc} \{T(A)\} & \longrightarrow & \{T(B)\} \\ \downarrow{\scriptstyle \pi_{T(A)}} & & \downarrow{\scriptstyle \pi_{T(B)}} \\ \{WP(A)\} & \xrightarrow[\{WP.(-)^*(g)\}]{} & \{WP(B)\} \end{array}$$