THE LEGACY OF
VLADIMIR VOEVODSKY

HOMOTOPY TYPES & RESIZING RULES

A FRESH LOOK AT THE IMPREDICATIVE SORT OF CIC

NICOLAS TABAREAU

Road Map

In this talk, I will recall two notions introduced by V.V. in 2006 in "*A very short note on homotopy λ-calculus*".

1. Homotopy types in type theory
2. Universe resizing rules

I will then explain how those two notions allow for a fresh look at the impredicative universe of CIC.

A Hierarchy of Types

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One of the main contribution of V.V. in type theory is the notion of levels of homotopy of types.

A Hierarchy of Types

Types are classified by the complexity of their equality/identity type.

Simplest (singleton) types are called contractible:

$$\mathsf{isContr}(A) :\equiv \sum_{(a:A)} \prod_{(x:A)} (a = x).$$

A Hierarchy of Types

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Proposition have a contractible equality:

$$\mathsf{isProp}(P) :\equiv \prod_{x,y:P} (x = y).$$

A Hierarchy of Types

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Then, n-Types are defined inductively:

Define the predicate is-n-type : $\mathcal{U} \to \mathcal{U}$ for $n \geq -2$ by recursion as follows:

$$\text{is-}n\text{-type}(X) :\equiv \begin{cases} \text{isContr}(X) & \text{if } n = -2, \\ \prod_{(x,y:X)} \text{is-}n'\text{-type}(x =_X y) & \text{if } n = n' + 1. \end{cases}$$

A Hierarchy of Types

This defines the following hierarchy:

Level of Type	Homotopy Type Theory
(-2)-Type	unit / contactible type
(-1)-Type	h-propositions
0-Type	h-sets
1-Type	h-groupoids
…	…
Type	∞-groupoids

A Hierarchy of Universes

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To avoid paradox à la Russell, we need to introduce a hierarchy of universes in type theory.

$$\vdash U_i : U_{i+1}$$

A Hierarchy of Universes

This is a sufficient condition to ensure consistency, but it is often a bit overkilled and one would like to relax it.

A Hierarchy of Universes

Syntactically, the management of the hierarchy can be improved by universe polymorphism which allows to use the same definition at different levels.

A Hierarchy of Universes

V.V. has proposed a semantic way to relax the hierarchy, based on so-called resizing rules.

Resizing Rules

Resizing rule for equivalent types.

$$(RR5) \quad \frac{U : Univ \qquad \Gamma \vdash X_1 : U \qquad \Gamma \vdash is : weq\, X_1\, X_2}{\Gamma \vdash X_2 : U}$$

(from V.V. 's talk at Bergen, 2011)

Resizing Rules

In a classical setting, every mere proposition is equivalent to either True or False.

True and False can be typed in the lowest universe.

Resizing Rules

Resizing rule for mere propositions.

$$\textbf{RR1} \qquad \frac{\Gamma \vdash is : isaprop\, X}{\Gamma \vdash X : UU}$$

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This is corresponds to the impredicativity of Prop

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$$x =_A y$$

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This means that the current Prop is implicitly assuming that every type is an h-set !

A Fresh Look at Prop

One possible way out
(as done in the HoTT Coq library):

Treat Prop as a taboo and not use it.

A Fresh Look at Prop

But maybe we can do better and fix it ?

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The rest of this talk is joint work with Gaetan Gilbert and Matthieu Sozeau.

Gaetan is implementing this feature, to be integrated hopefully in a future version Coq.

Prop under the Knife of HoTT

When an inductive type is defined in Prop, it can be eliminated only when building a Prop.

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This corresponds to the fact that propositional truncation can be eliminated

$$(A \to B) \to (||A|| \to B)$$

only when B is a mere proposition.

Prop under the Knife of HoTT

First motto:

> "Defining an inductive type in Prop corresponds to using propositional truncation"

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That is, morally, every type in Prop is squashed.

When Props produce Types

In CIC, there is the so-called singleton elimination:

> "A singleton definition has only one constructor and all the arguments of this constructor have type Prop."

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This covers for instance conjunction or the accessibility predicate but also equality !

When Props produce Types

With this new insight, singleton elimination can be seen as a syntactic condition on P:Prop which ensures that

$$||P|| \cong P$$

Problem

Allowing squashed equality to be unsquashed
is implicitly assuming that every type is an h-set

UIP hard-coded

Problem

The problem is that it doesn't take into account the number of occurrences of parameters/arguments in the return type.

When Props produce Types (II)

```
Inductive eq (A:Type)(x:A): A -> Prop
:= eq_refl : eq A x x.
```

a variable that occurs twice must be in h-sets.

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What about functions occurring in the return type ?

```
Vect (A : Prop) : nat -> Prop :=
  nil  : Vect A 0
| cons : A -> forall n : nat,
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`S` **must be injective**

What about multiple constructors ?

```
Inductive le : nat -> nat -> Prop :=
   le_O : forall n : nat, O <= n
|  le_S : forall n m : nat, m <= n -> S m <= S n
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Sums don't preserve mere propositions in general, but they do for disjoint sums.

the return types of different constructors must be orthogonal

Remark
Definitions Matter

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Inductive le' (n : nat) : nat -> Prop :=
    le_n : n <= n
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the criterion does not work for this (equivalent) definition

When a Prop is h-Prop

1. every argument that does not appear in the return type must be in Prop
2. every argument/parameters that appears more than once in the return type must be h-Set
3. every argument that appears exactly once is OK
4. the return types of different constructors must be orthogonal

When a Prop is -1-Type

1. every argument that does not appear
   in the return type must be in -1-Type
2. every argument/parameters that appears
   more than once in the return type must be 0-Type
3. every argument that appears exactly once is OK
4. the return types of different constructors
   must be orthogonal

Going to Higher Level

This characterisation generalises to n-types

1. every argument that does not appear
   in the return type must be in n-Type
2. every argument/parameters that appears
   more than once in the return type must be (n+1)-Type
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only for mere proposition

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This characterisation is very similar to what Jesper Cockx et al. use to do pattern-matching without K in Agda.

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We will be working in February with Jesper (thanks to EUTypes STSMs) to extend it .

What is this Impredicative Universe ?

The least we get is a new version of Coq:

- compatible with UIP
- compatible with univalence
- admitting the axiom :

forall (P:Prop) (x y : P), x = y

We Want More !

We Want More !

Replace the admissible axiom with a definitional equality:

$$\text{forall (P:Prop) (x y : P), } x \equiv y$$

Problem

Congruence with pattern-matching and fixpoints requires to apply inversion lemma even to neutral terms … and this potentially infinitely many times.

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Congruence with pattern-matching and fixpoints requires to apply inversion lemma even to neutral terms … and this potentially infinitely many times.

A naive implementation gives rise to an undecidable type checker !

Acc is a Hack

Perfectly valid mere proposition,
but with infinite unfolding …

```
Inductive Acc (A : Type) (R : A -> A -> Prop) (x : A) : Prop :=
  Acc_intro : (forall y : A, R y x -> Acc R y) -> Acc R x
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Definition Acc_inv : Acc R x -> forall y:A, R y x -> Acc R y.
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```
Definition Acc_inv : Acc R x -> forall y:A, R y x -> Acc R y.
```

a ≡ Acc_intro x (Acc_inv a) ≡ Acc_intro x (Acc_inv ...)

Acc is a Hack

It is not possible to guess how many times an inhabitant of Acc R x has to be unfolded.

Termination-unfolding criterion

We need to enforce termination of inversion through a syntactic check similar to the guard condition for fixpoints.

That is, recursive arguments of a constructor must have as indices strict sub terms of the indices of the return type.

Examples

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Remark

This syntactic characterisation of mere propositions is incomplete as for instance singleton types are not accepted.

This is somehow a good point because allowing singleton types in a definitional proof-irrelevant universe implies UIP *(Peter L.L.)*.

The Big Picture

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SProp — Impredicative

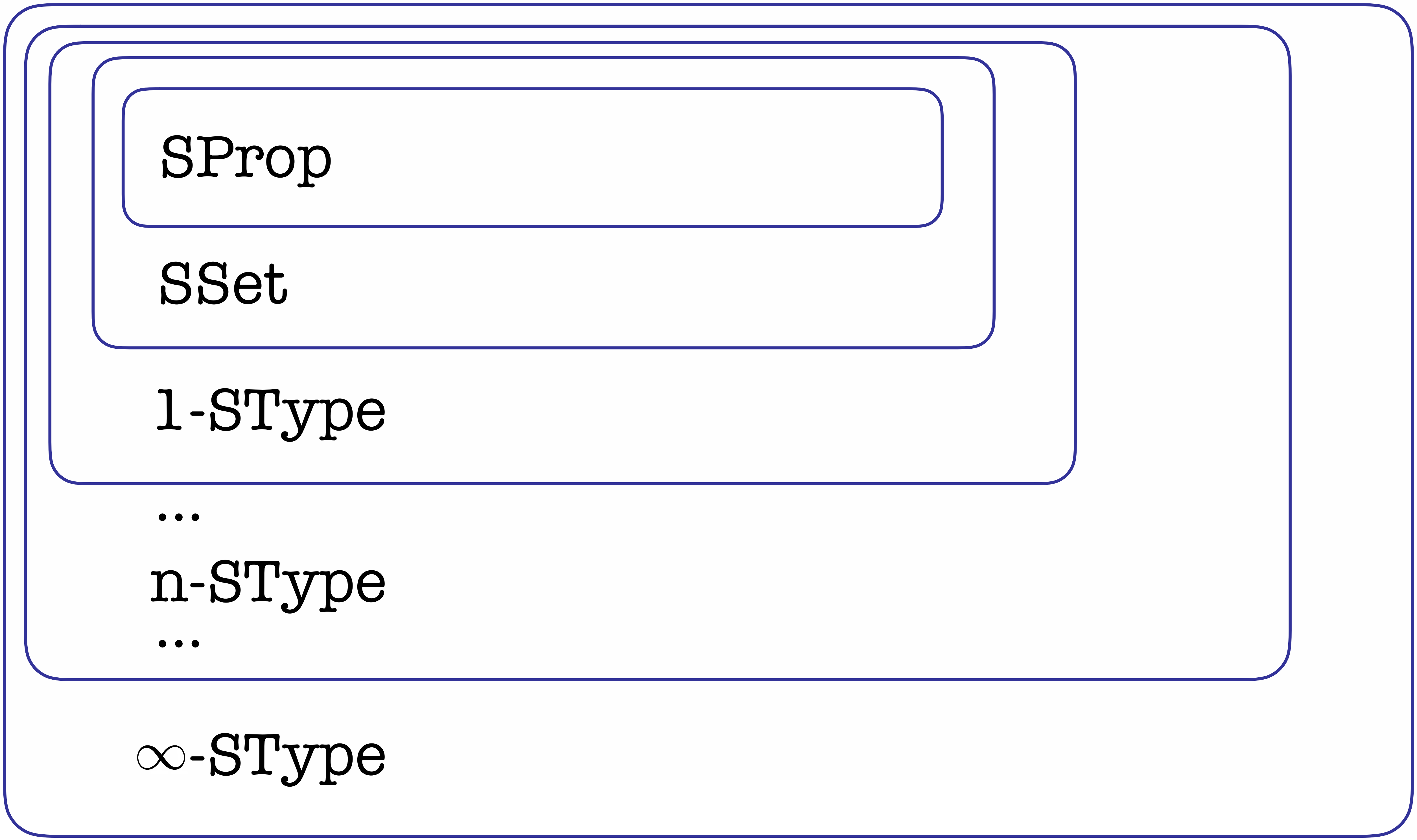
forall (P:Prop) (x y : P), x ≡ y

Prop — Impredicative

forall (P:Prop) (x y : P), x = y

Type — Predicative

Getting High(er) ?



V.V. has already sketched this in 2006!

$$\begin{array}{ccccccccc}
U_{0,0} & = & U_{1,0} & = & U_{2,0} & = & U_{3,0} & = & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & U_{1,1} & \longrightarrow & U_{2,1} & \longrightarrow & U_{3,1} & \longrightarrow & \dots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U_{2,2} & \longrightarrow & U_{3,2} & \longrightarrow & \dots \\
 & & & & & & \downarrow & & \\
 & & & & & & U_{3,3} & \longrightarrow & \dots
\end{array}$$

A very short note on homotopy λ-calculus
Vladimir Voevodsky, 2006

Demo

Doggy bag

1. Prop can be turned into a syntactic approximation of mere propositions
2. To get definitional proof-irrelevance, we also need to restrict recursive types with a guard condition
3. This should be (hopefully) available soon in Coq
4. It may be extended to deal with a // hierarchy of universes that encodes for homotopy levels.