Designing Dijkstra Monads

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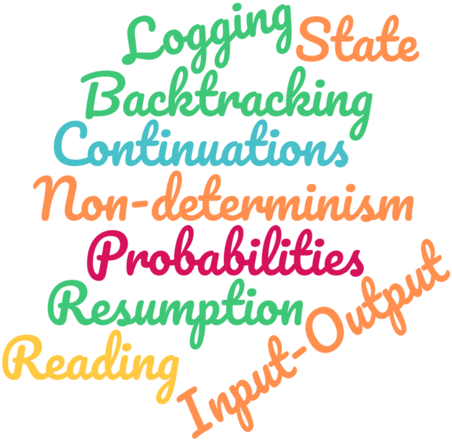
Programming with side effects



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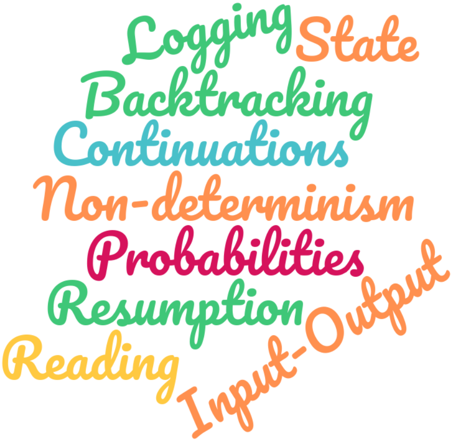

We need a model to uniformly account for side-effects !

Monads: a nifty algebraic tool

$\mathbb{M}X$ represents a computational context producing values in X

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A succint syntactic presentation: a type constructor M

$$\mathtt{ret}^{\mathrm{M}} : X \to \mathrm{M}X \qquad \mathtt{bind}^{\mathrm{M}} : \mathrm{M}X \to (X \to \mathrm{M}Y) \to \mathrm{M}Y$$

$+$ 3 equations (monad laws)

Examples of computational monads

State

$\mathrm{St}\, X = \mathcal{S} \to X \times \mathcal{S}$

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INPUT-OUTPUT
$\mathrm{IO}\, X = \mu Z. X + (Z \times \mathcal{O}) + (\mathcal{I} \to Z)$

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$\mathrm{Cont}_{\mathrm{Ans}}\,X = (X \to \mathrm{Ans}) \to \mathrm{Ans}$

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$\mathrm{Cont}_{\mathrm{Ans}}\,X = (X \to \mathrm{Ans}) \to \mathrm{Ans}$

NON-DETERMINISM
$\mathrm{NDet}\,X = \mathcal{P}_{fin}(X)$

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Non-determinism
$\mathrm{NDet}\,X = \mathcal{P}_{fin}(X)$

Probabilities
$\mathrm{Prob}\,X = \{\ f : X \to [0;1] \ \mid\ \mathrm{supp}(f) \text{ finite}, \Sigma_{x \in \mathrm{supp}(f)} f\,x \leq 1\ \}$

Monadic programming

Moggi's computational lambda-calculus (λ_C)(Moggi, 1989)

$$
\begin{aligned}
T &::= \mathrm{M}\,T \mid T_1 \to T_2 \mid T_1 \times T_2 \mid T_1 + T_2 && \text{types} \\
v &::= x \mid \lambda x.\, c \mid \langle c_1, c_2 \rangle \mid \iota_i\ v && \text{values} \\
c &::= v && \text{returning values} \\
&\quad \mid \texttt{let}\, x = c_1\, \texttt{in}\, c_2 && \text{sequencing computations} \\
&\quad \mid \texttt{op}(c_1, \ldots, c_n) && \text{effectful operations} \\
&\quad \mid c_1\ c_2 \mid \pi_i\ c \mid \texttt{case}\ c\ x.c_1\ y.c_2
\end{aligned}
$$

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Ret

$$\frac{\Gamma \vdash_{val} v : T}{\Gamma \vdash v : \mathrm{M}\ T}$$

Bind

$$\frac{\Gamma \vdash c_1 : \mathrm{M}\ T_1 \qquad \Gamma, x : T_1 \vdash c_2 : \mathrm{M}\ T_2}{\Gamma \vdash \texttt{let}\ x = c_1\ \texttt{in}\ c_2 : \mathrm{M}\ T_2}$$

Operations associated to monads

a.k.a. **generic effects** (Plotkin and Power, 2003)

STATE

$\texttt{get} : \mathbb{1} \to \mathrm{St}\,\mathcal{S}$

$\texttt{put} : \mathcal{S} \to \mathrm{St}\,\mathbb{1}$

INPUT-OUTPUT

$\texttt{read} : \mathbb{1} \to \mathrm{IO}\,\mathcal{I}$

$\texttt{write} : \mathcal{O} \to \mathrm{IO}\,\mathbb{1}$

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Exceptions

$\mathtt{throw} : \mathcal{E} \to \mathrm{Exc}\,\mathbb{0}$

Partiality

$\mathtt{div} : \mathbb{1} \to \mathrm{Div}\,\mathbb{0}$

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PARTIALITY

$\texttt{div} : \mathbb{1} \to \mathrm{Div}\,\mathbb{0}$

NON-DETERMINISM

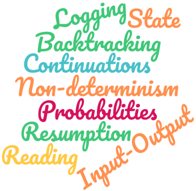
$\texttt{choose} : \mathbb{1} \to \mathrm{NDet}\,\mathbb{B}$

$\texttt{fail} : \mathbb{1} \to \mathrm{NDet}\,\mathbb{0}$

PROBABILITIES

$\texttt{flip} : [0;1] \to \mathrm{Prob}\,\mathbb{B}$

Starting point

Fix M a computational monad $\in$ 

Take a program $c : \mathrm{M}\mathbb{N}$, for instance

```
let x = read () in
let y = read () in
if y = 0 then throw Div_by_zero else x/y
```

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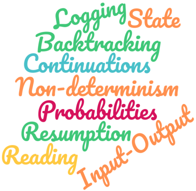

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How can we specify and verify such monadic programs ?

Hoare logic Primer

$$\{\ pre\ \}\ code\ \{\ post\ \}$$

$$\{\ \top\ \}\ v\ \{\ x = v\ \}$$

$$\frac{\{\ P\ \}\ c_1\ \{\ Q\ \} \qquad \forall x, \{\ Q(x)\ \}\ c_2\ \{\ R\ \}}{\{\ P\ \}\ \texttt{let}\ x = c_1\ \texttt{in}\ c_2\ \{\ R\ \}}$$

Hoare logic Primer

$$\{\, pre \,\}\ code\ \{\, post \,\}$$

$$\{\, \top \,\}\ v\ \{\, x = v \,\} \qquad \frac{\{\, P \,\}\ c_1\ \{\, Q \,\} \qquad \forall x, \{\, Q(x) \,\}\ c_2\ \{\, R \,\}}{\{\, P \,\}\ \texttt{let}\ x = c_1\ \texttt{in}\ c_2\ \{\, R \,\}}$$

Pure:	$pre : \mathbb{P}$	$post : X \to \mathbb{P}$
Stateful:	$pre : S \to \mathbb{P}$	$post : X \times S \to \mathbb{P}$
With exceptions:	$pre : \mathbb{P}$	$post : X + E \to \mathbb{P}$

Weakest precondition calculi

Dijkstra's insight: there is a **weakest** precondition that can be computed from a program and a postcondition

$$\vdash \{ P \} \; c \; \{ Q \} \qquad \Longleftrightarrow \qquad \vdash P \Rightarrow \mathtt{wp}[c](Q)$$

$$\mathtt{wp}[v](Q) = Q(v) \qquad \mathtt{wp}[\mathtt{let}\, x = c_1 \,\mathtt{in}\, c_2](Q) = \mathtt{wp}[c_1](\lambda x.\, \mathtt{wp}[c_2](Q))$$

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Pure: $\mathtt{wp}[c](\cdot) : (X \to \mathbb{P}) \to \mathbb{P}$

Stateful: $\mathtt{wp}[c](\cdot) : (X \times S \to \mathbb{P}) \to S \to \mathbb{P}$

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Wait. . . That's a monad !

Pure: $\mathrm{W}^{\mathrm{Id}} : (X \to \mathbb{P}) \to \mathbb{P}$

$$\mathtt{ret}^{\mathrm{W}^{\mathrm{Id}}} : X \to \mathrm{W}^{\mathrm{Id}} X \quad \mathtt{bind}^{\mathrm{W}^{\mathrm{Id}}} : \mathrm{W}^{\mathrm{Id}} X \to (X \to \mathrm{W}^{\mathrm{Id}} Y) \to \mathrm{W}^{\mathrm{Id}} Y$$

$$\mathtt{ret}^{\mathrm{W}^{\mathrm{Id}}}\ x\ Q = Q(x) \qquad \mathtt{bind}^{\mathrm{W}^{\mathrm{Id}}}\ w_1\ w_2\ Q = w_1(\lambda x.\, w_2(x)(Q))$$

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Continuation monad with answer type $\mathbb{P}$:

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Specification monads

Key idea: **specifications** can also be uniformly captured by monads!

Weakest precondition: $\mathrm{Cont}_{\mathbb{P}}\, X = (X \to \mathbb{P}) \to \mathbb{P}$

Strongest postcondition: $\mathrm{StrPost}\, X = \mathbb{P} \to X \to \mathbb{P}$

Pre/Postconditions: $\mathrm{PrePost}\, X = \mathbb{P} \times (X \to \mathbb{P})$

Specification monads

Key idea: **specifications** can also be uniformly captured by monads!

Weakest precondition:	$\mathrm{Cont}_{\mathbb{P}}\, X = (X \to \mathbb{P}) \to \mathbb{P}$
Strongest postcondition:	$\mathrm{StrPost}\, X = \mathbb{P} \to X \to \mathbb{P}$
Pre/Postconditions:	$\mathrm{PrePost}\, X = \mathbb{P} \times (X \to \mathbb{P})$

What's in a specification monad W ?

- specifications can be compared by strength

$$w_1 \leq^{\mathrm{Cont}_{\mathbb{P}}\, X} w_2 \qquad \Leftrightarrow \qquad \forall p : X \to \mathbb{P}, w_1\, p \implies w_2\, p$$

- $\texttt{bind}^{\mathrm{W}}$ is monotonic in both its arguments

$\rightsquigarrow$ restriction to monotonic elements in $\mathrm{Cont}_{\mathbb{P}}, \mathrm{StrPost}$

Predicate transformers from monad transformers

Examples of **predicate transformer** monads:

Pure: $\mathrm{W}^{\mathrm{Id}} : (X \to \mathbb{P}) \to \mathbb{P}$

Stateful: $\mathrm{W}^{\mathrm{St}} : (X \times \mathcal{S} \to \mathbb{P}) \to \mathcal{S} \to \mathbb{P}$

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$$\mathrm{W}^{\mathrm{Id}} = \mathcal{T}^{\mathrm{Id}}(\mathrm{Cont}_{\mathbb{P}}) \qquad \mathcal{T}^{\mathrm{Id}}(\mathrm{M}) = \mathrm{M}$$

$$\mathrm{W}^{\mathrm{St}} = \mathcal{T}^{\mathrm{St}}(\mathrm{Cont}_{\mathbb{P}}) \qquad \mathcal{T}^{\mathrm{St}}(\mathrm{M}) = \mathcal{S} \to \mathrm{M}(- \times \mathcal{S})$$

$$\mathrm{W}^{\mathrm{Exc}} = \mathcal{T}^{\mathrm{Exc}}(\mathrm{Cont}_{\mathbb{P}}) \qquad \mathcal{T}^{\mathrm{Exc}}(\mathrm{M}) = \mathrm{M}(- + \mathcal{E})$$

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Monad transformer $\mathcal{T}$:

- map a monad M to a monad $\mathcal{T}\mathrm{M}$
- $\texttt{lift}^{\mathcal{T}} : \mathrm{M} \to \mathcal{T}\mathrm{M}$

Relating computations & specifications

$\mathbb{M}$

computational monad

code

$c : \mathbb{M}\ \mathbb{N}$

$\mathbb{W}$

specification monad

specification

$w_c : \mathbb{W}\ \mathbb{N}$

Relating computations & specifications

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$w_c : \mathbb{W}\ \mathbb{N}$

- ▷ For a fixed $\mathbb{M}$, is there a canonical $\mathbb{W}$?

Relating computations & specifications

$$\mathrm{M} \xleftrightarrow{\theta} \mathrm{W}$$

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- For a fixed M, is there a canonical W ?
- What kind of structure θ relate M and W ?

Relating computations & specifications

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computational monad

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specification

$w_c : \mathrm{W}\ \mathbb{N}$

- For a fixed M, is there a canonical W ?
- What kind of structure θ relate M and W ?
- Is θ canonical wrt. M and W ?

Specifying programs with exceptions

$$\theta^{\mathrm{Exc}} \quad : \quad \mathrm{Exc}\, X \quad \longrightarrow \quad \mathrm{W}^{\mathrm{Exc}}\, X = (X + \mathcal{E} \to \mathbb{P}) \to \mathbb{P}$$

$$\theta^{\mathrm{Exc}}(m) = \lambda p.\, p\, m$$

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$$\theta^{\mathrm{Exc}}(\mathtt{ret}^{\mathrm{Exc}}\, v) = \mathtt{ret}^{\mathrm{W}^{\mathrm{Exc}}}\, v$$

$$\theta^{\mathrm{Exc}}(\mathtt{bind}^{\mathrm{Exc}}\, m\ f) = \mathtt{bind}^{\mathrm{W}^{\mathrm{Exc}}}\, (\theta^{\mathrm{Exc}}\, m)\ (\theta^{\mathrm{Exc}} \circ f)$$

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$$\theta^{\mathrm{Tot}} \quad : \quad \mathrm{Exc}\, X \quad \longrightarrow \quad \mathrm{W}^{\mathrm{Id}}\, X = (X \to \mathbb{P}) \to \mathbb{P}$$

$$\theta^{\mathrm{Tot}}(\mathtt{inr}\, v) = \lambda p.\, p\, v \qquad\qquad \theta^{\mathrm{Tot}}(\mathtt{inl}\, e) = \lambda p.\, \bot$$

Specifying programs with exceptions

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$$\theta^{\mathrm{Part}} \quad : \quad \mathrm{Exc}\, X \quad \longrightarrow \quad \mathrm{W}^{\mathrm{Id}}\, X = (X \to \mathbb{P}) \to \mathbb{P}$$

$$\theta^{\mathrm{Part}}(\mathtt{inr}\, v) = \lambda p.\, p\, v \qquad \theta^{\mathrm{Part}}(\mathtt{inl}\, e) = \lambda p.\, \top$$

Effect observation

An effect observation θ (Katsumata, 2014)

$$\mathrm{M} \xrightarrow{\quad\theta\quad} \mathrm{W}$$

is a **monad morphism**, that is

$$\theta(\mathtt{ret}^{\mathrm{M}}\ v) = \mathtt{ret}^{\mathrm{W}}\ v \qquad \theta(\mathtt{bind}^{\mathrm{M}}\ m\, f) = \mathtt{bind}^{\mathrm{W}}\ (\theta\ m)\ (\theta \circ f)$$

Relating stateful computations & specifications

$$\theta^{\mathrm{St}} : \begin{cases} \mathcal{S} \to X \times \mathcal{S} & \longrightarrow & \mathrm{W}^{\mathrm{St}}\, X = (X \times \mathcal{S} \to \mathbb{P}) \to \mathcal{S} \to \mathbb{P} \\ m & \longmapsto & \lambda p\, s_0.\, \texttt{let}\ \langle r,\ s_1 \rangle = m\, s_0\ \texttt{in}\ p\ \langle r,\ s_1 \rangle \end{cases}$$

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$$\frac{\mathrm{Id} \xrightarrow{\quad\mathtt{ret}\quad} \mathrm{Cont}_{\mathbb{P}}}{\theta^{\mathrm{St}} : \mathcal{T}^{\mathrm{St}}(\mathrm{Id}) \xrightarrow{\quad\mathcal{T}^{\mathrm{St}}(\mathtt{ret})\quad} \mathcal{T}^{\mathrm{St}}(\mathrm{Cont}_{\mathbb{P}})}$$

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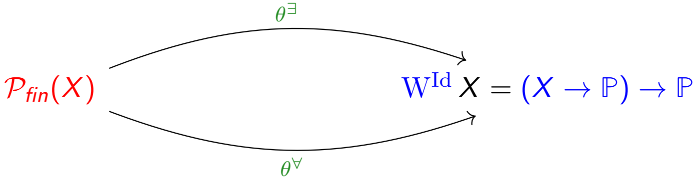
We can do that whenever we have a monad transformer $\mathcal{T}$

$$\mathcal{T}^{\mathrm{Exc}} \qquad \mathcal{T}^{\mathrm{St}} \circ \mathcal{T}^{\mathrm{Exc}} \qquad \mathcal{T}^{\mathrm{Exc}} \circ \mathcal{T}^{\mathrm{St}} \qquad \ldots$$

Verifying non-deterministic programs



$$\theta^{\exists}(\{v_1, \ldots, v_n\}) \quad = \quad \lambda p.\, p\, v_1 \vee \ldots \vee p\, v_n$$
$$\theta^{\forall}(\{v_1, \ldots, v_n\}) \quad = \quad \lambda p.\, p\, v_1 \wedge \ldots \wedge p\, v_n$$

Angelic non-determinism $\theta^{\exists}$ vs **Demonic** non-determinism $\theta^{\forall}$

Input-Output in context

$$\mathcal{L} = (\mathcal{I} + \mathcal{O})^*$$

$$\mathrm{IO}\,X \xrightarrow{\theta^{\mathrm{S}}} \mathrm{W}^{\mathrm{Id}}\,X = (X \to \mathbb{P}) \to \mathbb{P}$$

$$\mathrm{IO}\,X \xrightarrow{\theta^{\mathrm{Fr}}} \mathrm{W}^{\mathrm{Fr}}\,X = (X \times \mathcal{L} \to \mathbb{P}) \to \mathbb{P}$$

$$\mathrm{IO}\,X \xrightarrow{\theta^{\mathrm{Hist}}} \mathrm{W}^{\mathrm{Hist}}\,X = (X \times \mathcal{L} \to \mathbb{P}) \to \mathcal{L} \to \mathbb{P}$$

$$\theta^{\mathrm{S}}(\texttt{write}\,o) = \lambda p.\, p\,()$$

$$\theta^{\mathrm{Fr}}(\texttt{write}\,o) = \lambda p.\, p\,\langle (),\ [o] \rangle$$

$$\theta^{\mathrm{Hist}}(\texttt{write}\,o) = \lambda p\, log.\, p\,\langle (),\ (o :: log) \rangle$$

A second bridge between computations and specifications

M	W
computational monad	**predicate transformer** monad
code	specification
$c : \mathrm{M}\ \mathbb{N}$	$w_c : \mathrm{W}\ \mathbb{N}$

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Dijkstra monad

$c : \mathcal{D}^{\mathrm{M}}\ \mathbb{N}\ w_c$

A second bridge between computations and specifications

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code

$c : \mathrm{M}\ \mathbb{N}$

specification

$w_c : \mathrm{W}\ \mathbb{N}$

Dijkstra monad

$c : \mathcal{D}^{\mathrm{M}}\ \mathbb{N}\ w_c$

$\texttt{ret}^{\mathcal{D}^{\mathrm{M}}} : (x : A) \to \mathcal{D}^{\mathrm{M}}\ A\ (\texttt{ret}^{\mathrm{W}}\ x)$

$$\frac{m : \mathcal{D}^{\mathrm{M}}\ A\ w_1 \qquad f : (x : A) \to \mathcal{D}^{\mathrm{M}}\ B\ w_2(x)}{\texttt{bind}^{\mathcal{D}^{\mathrm{M}}}\ m\ f : \mathcal{D}^{\mathrm{M}}\ B\ (\texttt{bind}^{\mathrm{W}}\ w_1\ w_2)}$$

Monadic program verification in F*

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```
let rec fib (n:ℕ)
  : PURE ℕ (λ p → ∀ r. r ≥ n ∧ r > 0 ⟹ p r)
  =
  if n ≤ 1 then 1 else fib (n-1) + fib (n-2)
```

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```
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```

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$\rightsquigarrow$ State, exceptions, State+Exceptions,...
$\rightsquigarrow$ No Input-output, non-determinism, probabilities...

From effect observation to Dijkstra monad

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$$\mathcal{D}^{\mathrm{M}} A\,(w : \mathrm{W}\,A) \quad = \quad \{m : \mathrm{M}\,A \mid w \leq^{\mathrm{W}} \theta(m)\}$$

A Dijkstra monad for demonic non-determinism

Recall that there is an effect observation

$$\theta^{\forall} \quad : \quad \mathrm{NDet}\, A \quad \longrightarrow \quad \mathrm{W}^{\mathrm{Id}}\, A = (A \to \mathbb{P}) \to \mathbb{P}$$

$$\theta^{\forall}(m) = \lambda p.\, \forall v \in m, p\, v$$

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We obtain a Dijkstra monad

$$\begin{aligned} \mathrm{ND} &: (A : \mathrm{Type}) \to (w : \mathrm{W}^{\mathrm{Id}}\, A) \to \mathrm{Type} \\ \texttt{choose} &: (u : \mathbb{1}) \to \mathrm{ND}\ \mathbb{B}\ (\lambda p.\, p\ \texttt{true} \wedge p\ \texttt{false}) \\ \texttt{fail} &: (u : \mathbb{1}) \to \mathrm{ND}\ \mathbb{B}\ (\lambda p.\, \top) \end{aligned}$$

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```
let rec pick #a (l : list a)
  : ND a (λ p → ∀ x. List.memP x l ⟹ p x)
  =
  match l with
  | [] → fail ()
  | x∷xs → if choose true false then x else pick xs
```

Computing pythagorean triples

```
let guard (b:bool)
  : ND unit (λ p → b ⟹ p ())
  =
  if b then () else fail ()

let pyths ()
  : ND (ℤ & ℤ & ℤ)
       (λ p → ∀ x y z. x×x + y×y == z×z ⟹ p (x,y,z))
  =
  let l = [1;2;3;4;5;6;7;8;9;10] in
  let x = pick l in
  let y = pick l in
  let z = pick l in
  guard (x×x + y×y = z×z);
  (x,y,z)
```

Synthesis

- ▷ Start with a **computational monad** M
- ▷ Select a **specification monad** W
- ▷ Define an **effect observation** $\theta : \mathrm{M} \to \mathrm{W}$
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Further questions:

- ▷ How far is $\mathcal{D}^{\mathrm{M}}$ from θ ?
- ▷ Can we explain the previous awkward restrictions in DM4Free ?

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Extends to a (categorical) equivalence

Dijkstra monads	$\cong$	Effect observations
$(\mathrm{W}, \mathcal{D})$		$\theta : \mathrm{M} \to \mathrm{W}$

A DSL for monad transformers

$$C ::= \mathbb{M}A \mid C_1 \times C_2 \mid (x : A) \to C \mid C_1 \to C_2 \quad A \in Type_{\mathcal{L}}$$

$$t ::= \texttt{ret} \mid \texttt{bind} \mid \langle t_1,\ t_2 \rangle \mid \pi_i\ t \mid x \mid \lambda x.\, t \mid t_1\ t_2 \mid \lambda^{\diamond} x.\ t \mid t\ u$$

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Observation 1: from monads $C[X]$ and M we can derive a monad $\mathcal{T}^C(\mathrm{M})$
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$$\texttt{lift} \quad : \quad \mathrm{M} \xrightarrow{\mathrm{M}(\texttt{ret}^{\mathcal{T}^{C}(\mathrm{M})})} \mathrm{M}(\mathcal{T}^{C}(\mathrm{M})) \xrightarrow{\alpha} \mathcal{T}^{C}(\mathrm{M})$$

Reinterpreting DM4Free

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For a general monad C: $\mathcal{T}^C$ only preserves **monadic relations** not monad morphisms

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Thank you !

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