Efficient Mendler-Style Lambda-Encodings in Cedille

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cNat ◀ ⋆ = ∀ X : ⋆. (X → X) → X → X.

cZ ◀ cNat = Λ X. λ s. λ z. z.
cS ◀ cNat → cNat = λ n. Λ X. λ s. λ z. s (n s z).
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  cZ ◀ cNat = Λ X. λ s. λ z. z.
  cS ◀ cNat $\to$ cNat = λ n. Λ X. λ s. λ z. s (n s z).
- Essentially, we identify each natural number n with its iterator λ s. λ z. s^n z.

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- As a consequence, most languages come with built-in infrastructure for defining inductive datatypes (data definition, pattern-matching, termination checker, negativity and strictness check, etc.).

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data Nat : Set where
  zero : Nat
  suc  : Nat → Nat
```

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pred : Nat -> Nat
pred zero = zero
pred (suc n) = n
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- In Agda, induction principle can be derived by pattern matching and explicit structural recursion.

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  2. implicit products,
  3. primitive heterogeneous equality.
- Cedille is an implementation of CDLE type theory (in Agda!).

Extension: Dependent intersection types

- Formation

$$\frac{\Gamma \vdash T : \star \quad \Gamma, x : T \vdash T' : \star}{\Gamma \vdash \iota\, x{:}T.\, T' : \star}$$

- Introduction

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : [t_1/x]T' \quad \Gamma \vdash p : t_1 \simeq t_2}{\Gamma \vdash [t_1, t_2\{p\}] : \iota\, x{:}T.\, T'}$$

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$$\begin{array}{lcl} |[t_1, t_2\{p\}]| & = & |t_1| \\ |t.1| & = & |t| \\ |t.2| & = & |t| \end{array}$$

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- In Cedille, objects are types and natural transformations are polymorphic functions:

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AlgM ◂ ⋆ → ⋆ = λ X : ⋆. ∀ R : ⋆. (R → X) → F R → X.
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  `foldM ◀ ∀ X : ⋆. AlgM X → FixM → X = <..>`
- Constructors are expressed as a Church-style algebra:

  `inM ◀ F FixM → FixM = λ v. λ alg. alg (foldM alg) v.`

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- Mendler-style proof-algebras

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- Mendler-style proof-algebras

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PrfAlgM ◀ Π A : ⋆. (A → ⋆) → (F A → A) → ⋆
 = λ A. λ Q. λ alg.
   ∀ R : ⋆. ∀ c : R → A. ∀ e : (Π r : R. c r ≃ r).
   (Π r : R. Q (c r)) →
   Π fr : F R.  Q (alg (fmap c fr)).
```

Mendler-style induction principle

- The collection of constructors of type `FixIndM` is expressed by Church-algebra

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- Cancellation law:

```
indHom ◀ ∀ Q palg x.
 induction palg (inFixInd x) ≃ palg (induction palg) x
 = Λ Q. Λ palg. Λ x. β.
```

- Can we define a a proof-algebra which erases to lambda term `λ x. λ y. y`?

Constant-time destructor

- `outAlgM ◀ PrfAlgM FixIndM (λ _. F FixIndM) inFixIndM`
  `= Λ R. Λ c. Λ e. λ x. λ y. [ y , c y { e y } ].2.`

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- Finally, we arrive at the generic constant-time linear-space destructor of inductive datatypes:

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- Since `outFixIndM` is constant-time then we get Lambek's Lemma as an easy consequence

  `lambek1 ◀ Π x: F FixInd. outFixIndM (inFixIndM x) ≃ x`
  `= λ x. β.`

  `lambek2 ◀ Π x: FixIndM. inFixIndM (outFixIndM x) ≃ x`
  `= λ x. induction (Λ R. Λ c. Λ e. λ ih. λ fr. β) x.`

Example: Natural numbers

- Natural numbers arise as least fixed point of a scheme `NF`

```
NF ◀ ⋆ → ⋆ = λ X : ⋆. Unit + X.

Nat ◀ ⋆ = FixIndM NF.
```

- Constructors

```
zero ◀ Nat         =      inFixIndM (in1 unit).
suc  ◀ Nat → Nat = λ n. inFixIndM (in2 n).
```

- Constructor `suc` has the following underlying lambda-term

```
suc n ≃ λ alg. (alg (λ f. (f alg)) (λ i. λ j. (j n))).
```

- Constant-time predecessor

```
pred ◀ Nat → Nat
 = λ n. case (outFixIndM n) (λ _. zero) (λ m. m).
```

Identity mappings instead of functors

- The described developments are well-justified for any functor

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Functor ◀ (⋆ → ⋆) → ⋆ = λ F.
  Σ fmap : ∀ X Y : ⋆. (X → Y) → F X → F Y.
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- Surprisingly, the construction can be easily generalized to the larger class of schemes we call **identity mappings**

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- Converse is not true

```
UneqPair ◀ ⋆ → ⋆ = λ X. Σ x₁ x₂ : X. x₁ ≠ x₂.
```

- Identity mappings induce a large class of datatypes (including infinitary and non-strictly positive datatypes).

There is more!

- We generically define course-of-value datatypes and implement dependent histomorphisms. We do this by defining a least fixed point of a coend of "negative" scheme.

  Lift ◀ $(\star \to \star) \to \star \to \star$ = λ F. λ X. F X $\times$ (X $\to$ F X).

  FixCoV ◀ $(\star \to \star) \to \star$ = λ F. FixIndM (Coend (Lift F)).
- In a similar way, we generically derive (small) inductive-recursive datatypes and derive the respective dependent elimination.

Thank you!