Degrees of Relatedness

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One Slide Summary

- Parametricity is about **relations**,
- Objects are **related** $\rightsquigarrow$ Specify to what **degree** i $(s \frown_i t)$,
- The **larger** the type, the **more degrees** are eligible,
- Describe **function behaviour** by saying how functions **influence degree** of relatedness,
- This explains
  - **parametricity:** $\text{flatten} : (\textbf{par} \mid X : \mathcal{U}) \to \text{Tree}\ X \to \text{List}\ X$
  - **ad hoc polymorphism:** $\text{lem} : (\textbf{hoc} \mid X : \mathcal{U}) \to X \uplus (X \to \text{Empty})$
  - **. irrelevance:** $[] : (\textbf{irr} \mid n : \mathbb{N}) \to \text{List}_n\ A$
  - **.. shape-irrelevance:** $\lambda n.\text{List}_n\ A : (\textbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}$
  - aspects of **unions, intersections, algebra, Prop, . . .**

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Theorem

$$(A \to B) \cong \left(\underbrace{\forall X.(X \to A)}_{\textit{For any representation } (X, r) \textit{ of } A} \to (X \to B) \right)$$

Proof:

($\to$) $h \mapsto \lambda X.\lambda r.h \circ r$.

($\leftarrow$) $g \mapsto g\ A\ \mathrm{id}$.

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(tgt) Prove: $g\ X\ r\ x = g\ A\ \mathrm{id}\ (r\ x)$.

Parametricity: Representation independence

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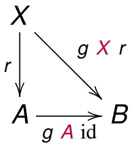

Relational parametricity

Lemma

If $g : \forall X.(X \to A) \to (X \to B)$
then $g\ X_0\ r_0\ x_0 = g\ A\ \mathrm{id}\ (r_0\ x_0)$.

Rel. param.: A sound scheme for proving parametricity theorems.
Idea: **Related things map to related things.**

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$$X : *, \qquad r : X \to A, \qquad x : X \qquad \vdash \qquad g\ X\ r\ x : B$$

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Π is not parametric

System F:
$\forall X.(X \to A) \to (X \to B)$.

Dependent types:
$\Pi(X : \mathcal{U}).(X \to A) \to (X \to B)$.

Suppose $B = \mathcal{U}$:
$leak : \Pi(X : \mathcal{U}).(X \to A) \to (X \to \mathcal{U})$
$leak\ X\ r\ x = X$.
Representation type is returned as data!

But think of $Leak\ X\ r\ x = X$ as a dependent type
We're just ignoring arguments

DTT: formal type/data boundary disappears
Difference in expectation remains

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Let's have two relations

System F

Values can be related:

$$(s : S) \frown (t : T)$$

IEL: if $(s : A) \frown (t : A)$ then $s = t$ (heterogeneous equality)

Types can be related:

$$S \frown T$$

which gives meaning to

$$(s : S) \frown (t : T)$$

Dependent types

Things can be 0-related:

$$(s : S) \frown_0 (t : T)$$

IEL: if $(s : A) \frown_0 (t : A)$ then $s = t$ (heterogeneous equality)

Things can be 1-related:

$$(s : K) \frown_1 (t : L)$$

where $(S : \mathcal{U}) \frown_1 (T : \mathcal{U})$ gives meaning to

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Let's have two relations

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Examples

0-relatedness:

- $(2+5 : \mathbb{N}) \frown_0 (7 : \mathbb{N})$
- $([\text{true}, \text{false}] : \text{List}_4\ \text{Bool}) \frown_0 ([\text{true}, \text{false}] : \text{List}_7\ \text{Bool})$
- $(\text{flatten Bool} : \text{Tree Bool} \to \text{List Bool}) \frown_0$
  $(\text{flatten}\ \mathbb{N} : \text{Tree}\ \mathbb{N} \to \text{List}\ \mathbb{N})$
  for any proof of $\text{Bool} \frown_1 \mathbb{N}$

1-relatedness:

- $(\text{List}_4\ \text{Bool} : \mathcal{U}) \frown_1 (\text{List}_7\ \text{Bool} : \mathcal{U})$
- $(\text{Bool} : \mathcal{U}) \frown_1 (\mathbb{N} : \mathcal{U})$ non-canonically
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Depth of a type determines **amount of relations**:

- Depth -1: Unit, Empty, $P \vee Q$, $\ldots$
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- Depth 0 (only equality): Bool, $\mathbb{N}$, List_n Bool, $\mathcal{U}^{-1}$, $\ldots$
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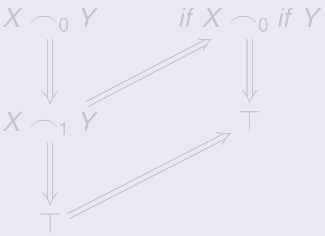
Understanding modalities (1)

par : $1 \to 0$

$$\mathit{if} : (\mathbf{par} \mid X : \mathcal{U}^0) \to \mathrm{Bool} \to X \to X \to X$$



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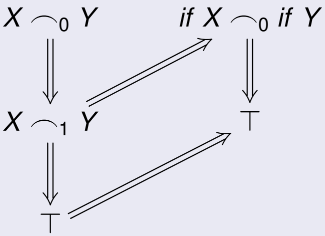

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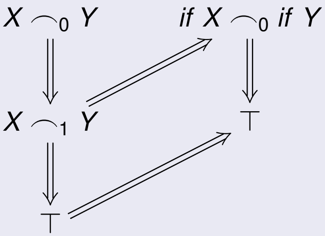

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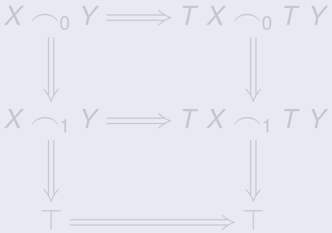

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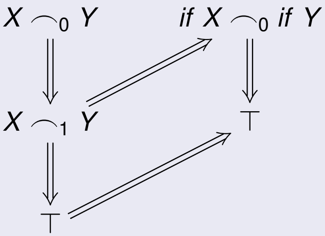

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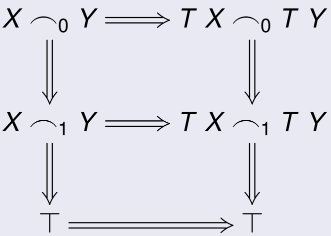

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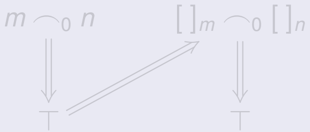

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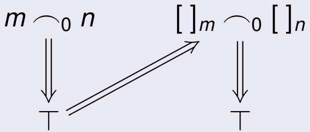

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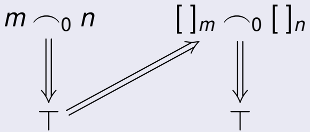

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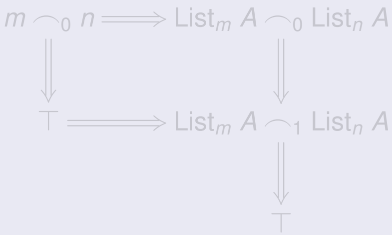

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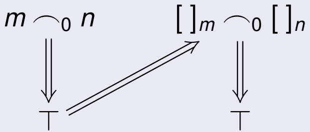

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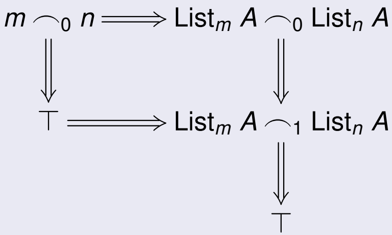

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$\mathit{if}\ (\mathrm{List}_n\ A)\ b\ \mathit{as}\ [\ ]_n$

Irrelevant in n?

Yes if $\mathbf{par} \circ \mathbf{shi} = \mathbf{irr} : 0 \to 0$

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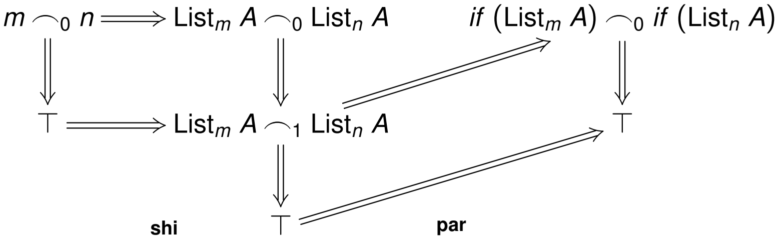

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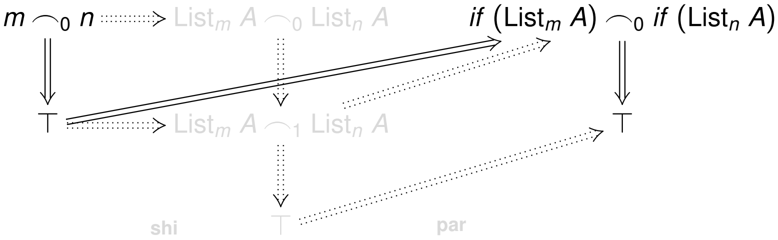

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  See Licata et al. (2016, 2017) for multi-mode type theory
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Thanks!

Questions?