

# Naturality for Free

The category interpretation of directed type theory

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# reverse is natural

List : **Set**  $\rightarrow$  **Set**

rev :  $\prod_{A:\mathbf{Set}} \text{List } A \rightarrow \text{List } A$

$f : A \rightarrow B$

$$\begin{array}{ccc} \text{List } A & \xrightarrow{\text{rev}_A} & \text{List } A \\ \text{List } f \downarrow & & \downarrow \text{List } f \\ \text{List } B & \xrightarrow{\text{rev}_B} & \text{List } B \end{array}$$

## Proof by ...-induction

$$\text{List } f [a_0, a_1, \dots, a_{n-1}] = [f a_0, f a_1, \dots, f a_{n-1}]$$

$$\text{rev}_A [a_0, a_1, \dots, a_{n-1}] = [a_{n-1}, \dots, a_1, a_0]$$

$$\begin{aligned} (\text{rev}_B \circ \text{List } f) [a_0, a_1, \dots, a_{n-1}] &= \text{rev}_B (\text{List } f [a_0, a_1, \dots, a_{n-1}]) \\ &= \text{rev}_B [f a_0, f a_1, \dots, f a_{n-1}] \\ &= [f a_{n-1}, \dots, f a_1, f a_0] \\ &= \text{List } f [a_{n-1}, \dots, a_1, a_0] \\ &= \text{List } f (\text{rev}_A [a_0, a_1, \dots, a_{n-1}]) \\ &= (\text{List } f \circ \text{rev}_A) [a_0, a_1, \dots, a_{n-1}] \end{aligned}$$

# Everything is natural ...

$F, G : \mathbf{Set} \rightarrow \mathbf{Set}$

$\alpha : \prod_{A:\mathbf{Set}} F A \rightarrow G A$

$f : A \rightarrow B$

$$\begin{array}{ccc} F A & \xrightarrow{\alpha_A} & G A \\ F f \downarrow & & \downarrow G f \\ F B & \xrightarrow{\alpha_B} & G B \end{array}$$

...but we can't prove it.

- We know that all families of functions are natural.
- But we cannot prove it.
- It should be a *free theorem*.

## The hint (HoTT)

$$F, G : \mathbf{Set} \rightarrow \mathbf{Set}$$

$$\alpha : \prod_{A:\mathbf{Set}} F A \simeq G A$$

$$f : A \simeq B$$

$$\begin{array}{ccc} F A & \xrightarrow{\alpha_A} & G A \\ F f \downarrow & & \downarrow G f \\ F B & \xrightarrow{\alpha_B} & G B \end{array}$$

- $A \simeq B$  means isomorphism (for sets).
- This is provable in HoTT.
- It follows from univalence + J.

# Summary

- The set-level fragment of HoTT can be interpreted using the groupoid model (Hofmann & Streicher).
- This interpretation also gives rise to a univalent, truncated universe of sets (but it doesn't classify hsets).
- Can we replace groupoids by categories?
- Yes, but we need to take care of polarities.
- And some places we do need groupoids, hence we need an operation calculating the groupoid associated to a category (the core).
- I am going to derive a type theory guided by the semantics.

# The category with families of categories

Contexts	$\text{Con} : \mathbf{Set}$	$\Gamma : \text{Con}$	$\llbracket \Gamma \rrbracket : \mathbf{Cat}$
Types	$\text{Ty} : \text{Con} \rightarrow \mathbf{Set}$	$A : \text{Ty } \Gamma$	$\llbracket A \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \mathbf{Cat}$
Terms	$\text{Tm} : (\Gamma : \text{Con}) \rightarrow \text{Ty } \Gamma \rightarrow \mathbf{Set}$	$a : \text{Tm } \Gamma A$	$\llbracket a \rrbracket : \dots$
Subst	$\text{Tms} : \text{Con} \rightarrow \text{Con} \rightarrow \mathbf{Set}$	$\gamma : \text{Tms } \Gamma \Delta$	$\llbracket \gamma \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \Delta \rrbracket$

$$\begin{array}{c}
 \llbracket \Gamma . A \rrbracket \\
 \downarrow \left. \begin{array}{c} \uparrow \\ \uparrow \end{array} \right) \llbracket a \rrbracket \\
 \llbracket \Gamma \rrbracket
 \end{array}$$



## Operations on contexts

$$\frac{}{\bullet : \text{Con}} \quad \frac{A : \text{Ty } \Gamma}{\Gamma.A : \text{Con}}$$

$$\begin{aligned} |\llbracket \bullet \rrbracket| &= \mathbf{1} \\ \llbracket \bullet \rrbracket(x, y) &= \mathbf{1} \end{aligned}$$

$$\begin{aligned} |\llbracket \Gamma.A \rrbracket| &= (x : |\llbracket \Gamma \rrbracket|) \times |\llbracket A \rrbracket x| \\ \llbracket \Gamma.A \rrbracket((x, a), (y, b)) &= (f : \llbracket \Gamma \rrbracket(x, y)) \times (\llbracket A \rrbracket y)(\llbracket A \rrbracket f a, b) \end{aligned}$$

*Grothendieck construction*

# Opposites

$$\frac{\Gamma : \mathbf{Con}}{\Gamma^{\text{op}} : \mathbf{Con}} \quad \frac{A : \mathbf{Ty} \Gamma}{A^{\text{op}} : \mathbf{Ty} \Gamma}$$

$$\begin{aligned} \llbracket \Gamma^{\text{op}} \rrbracket &= \llbracket \Gamma \rrbracket^{\text{op}} \\ \llbracket A^{\text{op}} \rrbracket x &= (\llbracket A \rrbracket x)^{\text{op}} \end{aligned}$$

- Note that  $_{\text{op}} : \mathbf{Cat} \rightarrow \mathbf{Cat}$  is covariant!
- But what is  $(\Gamma.A)^{\text{op}}$  ?
- It cannot be  $\Gamma^{\text{op}}.A^{\text{op}}$

# Opfibrations

$$\frac{A : \text{Ty } \Gamma^{\text{op}}}{\Gamma.^{\text{op}}A : \text{Con}}$$

$$\begin{aligned} |\llbracket \Gamma.^{\text{op}}A \rrbracket| &= (x : |\llbracket \Gamma \rrbracket|) \times |\llbracket A \rrbracket x| \\ \llbracket \Gamma.^{\text{op}}A \rrbracket((x, a), (y, b)) &= (f : \llbracket \Gamma \rrbracket(x, y)) \times (\llbracket A \rrbracket x)(a, \llbracket A \rrbracket f b) \end{aligned}$$

$$(\Gamma.A)^{\text{op}} = \Gamma^{\text{op}}.^{\text{op}}A^{\text{op}}$$

## $\Sigma$ -types, undirected

$$\frac{A : \text{Ty } \Gamma \quad B : \text{Ty } \Gamma.A}{\Sigma A B : \text{Ty } \Gamma}$$

$$\Gamma.A.B \cong \Gamma.(\Sigma A B)$$

On objects:

$$(\Sigma A B) x = (A x).(B x)$$

## $\Sigma$ -types with polarities

$$\frac{A : \mathbb{T}_y \Gamma^s \quad B : \mathbb{T}_y \Gamma.^s A}{\Sigma^s A B : \mathbb{T}_y \Gamma}$$

$$\Gamma.^t A.^s B \cong \Gamma.^t (\Sigma^s A B)$$

On objects:

$$(\Sigma^s A B)_x = (A_x).^s (B_x)$$

$$(\Sigma A B)^{\text{op}} = \Sigma^{\text{op}} A^{\text{op}} B^{\text{op}}$$

## $\Pi$ -types, undirected

$$\frac{A : \text{Ty } \Gamma \quad B : \text{Ty } \Gamma.A}{\Pi A B : \text{Ty } \Gamma}$$

$$\text{Ty } \Gamma.A B \cong \text{Ty } \Gamma (\Pi A B)$$

## $\Pi$ -types with polarities

$$\frac{A : \text{Ty } \Gamma^{\text{op}} \quad B : \text{Ty } \Gamma.{}^{\text{op}} A^s}{\Pi^s A B : \text{Ty } \Gamma}$$

$$\text{Trm } \Gamma.{}^{\text{op}} A^s B \cong \text{Trm } \Gamma (\Pi^s A B)$$

$$(\Pi A B)^{\text{op}} = \Pi^{\text{op}} A^{\text{op}} B^{\text{op}}$$

# The universe of sets

$$\frac{}{U : \text{Ty } \Gamma} \quad \frac{a : \text{Tm } \Gamma \ U}{\text{El } a : \text{Ty } \Gamma}$$

$$\begin{aligned} | \llbracket U \rrbracket x | &= \mathbf{Set} \\ (\llbracket U \rrbracket x)(A, B) &= A \rightarrow B \end{aligned}$$

$$\begin{aligned} | \llbracket \text{El } a \rrbracket x | &= \llbracket a \rrbracket x \\ (\llbracket \text{El } a \rrbracket x)(y, z) &= (y = z) \end{aligned}$$



# The hom type

$$\frac{A : \text{Ty } \Gamma}{- \sqsubseteq_A - : \text{Tm } \Gamma (A^{\text{op}} \rightarrow A \rightarrow \text{U})}$$

$$\llbracket a \sqsubseteq_A b \rrbracket x = Ax(ax, bx)$$

- But what about id (aka refl)?
- We would like to say

$$\text{id} : \prod a : A. a \sqsubseteq_A a$$

but this doesn't type check!

# The core type

$$\frac{A : \text{Ty } \Gamma}{\bar{A} : \text{Ty } \Gamma} \quad \frac{a : \text{Tm } \Gamma A}{\bar{a} : \text{Tm } \Gamma \bar{A}} \quad \frac{\begin{array}{l} f : \text{Tm } \Gamma (a \sqsubseteq_A b) \\ f^{\text{op}} : \text{Tm } \Gamma (b \sqsubseteq_A a) \\ l : \text{Tm } \Gamma (f \circ f^{\text{op}} \sqsubseteq \text{id}_a) \\ r : \text{Tm } \Gamma (f^{\text{op}} \circ f \sqsubseteq \text{id}_a) \end{array}}{f, f^{\text{op}}, l, r : \text{Tm } \Gamma (\bar{a} \sqsubseteq_{\bar{A}} \bar{b})}$$

$$\frac{a : \text{Tm } \Gamma \bar{A}}{\underline{a} : \text{Tm } \Gamma A^s} \quad \frac{f : \text{Tm } \Gamma (a \sqsubseteq_{\bar{A}} b)}{\underline{f} : \text{Tm } \Gamma (\underline{a} \sqsubseteq_A \underline{b})}$$

## Directed Path induction (J)

$$\text{id} : \text{Tm } \Gamma (\Pi a : \bar{A}. \underline{a} \sqsubseteq_A \underline{a})$$

$$M : \text{Tm } \Gamma (\Pi a : A^{\text{op}}, b : A. \underline{a} \sqsubseteq_A b \rightarrow \text{U})$$

$$m_{\text{id}} : \text{Tm } \Gamma (\Pi a : \bar{A}. M \underline{a} \underline{a} (\text{id } a))$$

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$$JM m_{\text{id}} : \text{Tm } \Gamma (\Pi a : A^{\text{op}}, b : A, f : a \sqsubseteq b. M a b f)$$

# The homtypes of sets

Homtypes of sets are symmetric

$$\frac{a : \text{Tm } \Gamma \ A}{\overline{\text{El } a} \simeq \text{El } a}$$

Homtypes of sets are proof irrelevant

$$\frac{a : \text{Tm } \Gamma \ A}{K_a : \text{Tm } \Gamma \ (\prod a : \bar{A}, p : \underline{a} \sqsubseteq_A \underline{a}.p \sqsubseteq \text{id } a)}$$

Directed univalence

$$\text{coe} : \text{Tm } \Gamma \ (\prod a, b : U. a \sqsubseteq_U b \rightarrow (\text{El } a \rightarrow \text{El } b))$$

coe is an isomorphism.

# Everything is natural, provably!

$F, G : \mathbf{Set} \rightarrow \mathbf{Set}$

$\alpha : \prod_{A:\mathbf{Set}} F A \rightarrow G A$

$f : A \rightarrow B$

$$\begin{array}{ccc} F A & \xrightarrow{\alpha_A} & G A \\ F f \downarrow & & \downarrow G f \\ F B & \xrightarrow{\alpha_B} & G B \end{array}$$

- It follows from directed univalence + directed J.

## Further work

- Filippo is formalizing the calculus and its semantics in Agda.
- What is the relation to logical relations?
- Can we do higher categories (full directed HoTT)?